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THE STEINER POINT IN INFINITE DIMENSIONS

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ABSTRACT
It is shown that the Steiner point cannot be extended continuously to all convex bodies in infinite dimensional Hilbert spaces. This follows as a corollary of a result on the local behavior of the point.

1. Introduction

The Steiner point of a convex body has been studied from a variety of points of view (see for instance, [5], [6], and [9] for references and discussion; also, [2]). Its many attractive properties and recent use in differential inclusions ([1]) and random sets ([3], [4], [15]) have raised the question of whether it can be extended continuously to convex bodies in infinite dimensions. The purpose of this note is to show that this unfortunately cannot be done. Using an argument with a probabilistic component, we actually show somewhat more, that in infinite dimensions the Steiner points of finite dimensional convex bodies neighboring a given body occupy the largest possible region.

2. Preliminaries

The setting will be a Hilbert space $\mathbf{H}$ with inner product $\langle \cdot , \cdot \rangle$ and collection $\mathbf{K}$ of convex bodies (non-empty compact, convex subsets) topologized by the Hausdorff metric ρ. To each $K \in \mathbf{K}$ is associated its *support function* h, given by $h(x,K) = \max\{\langle x, y\rangle \mid y \in K\}$, $x \in \mathbf{H}$. For $\mathbf{H} = \mathbf{E}^d$, the Steiner point of $K \in \mathbf{K}$ can be expressed as

$$(1) \qquad s(K) = d \int_{S^{d-1}} uh(u,K)\sigma(du)$$

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where S^{d-1} is the unit sphere equipped with unit Lebesgue measure σ ([13]).

We note that an immediate consequence of (1) is the Lipschitz continuity of the Steiner point. However, the Lipschitz constant grows like $\sqrt{d}$ with increasing dimension (see appendix). Together with the fact that the Steiner point of a convex body is independent of the Euclidean space in which the body is embedded (a consequence of its uniqueness, e.g. [10], [11], [12]), this implies that there can be no Lipschitz, or even uniformly, continuous extension to all convex bodies in infinite dimensions. Using other means, chiefly an appeal to [8] for a refinement of a lemma in [7], Giné and Hahn [4] have shown that a uniformly continuous extension is impossible even for centroids which lack the full range of properties of the Steiner point.

On the other hand, these comments do not resolve the question of a merely continuous extension. Indeed, there is an abundant supply of functionals on **K** (in finite or infinite dimensions) which are continuous but not uniformly continuous. A typical one is

$$f(K)=\sum_{j=1}^{\infty}\frac{1}{j}\left[h\left(\left(1-\frac{1}{j}\right)u_0+\frac{1}{j}u_1,K\right)-h(u_0,K)\right]$$

for orthogonal unit vectors u_0, u_1 (note that this functional also possesses the natural linearity property $f(\alpha K+\beta K')=\alpha f(K)+\beta f(K')$, $\alpha,\beta\geq 0$). Different continuity assumptions are also reflected in [10], [11], and [12]. Accordingly, to rule out a continuous extension of the Steiner point, another approach is needed.

3. No continuous extension

It will be useful to recast a traditional formulation of the Steiner point ([6]) in probabilistic terms: given a polytope $P=\operatorname{conv}\{v_1,\dots,v_r\}\subseteq \mathbf{E}^d$, note that, for σ-almost all hyperplanes with normal $u\in S^{d-1}$, there is a unique solution $x\in\{v_1,\dots,v_r\}$ to $\langle u,x\rangle=h(u,K)$. Call this point $v(u,P)$. If U is a random unit vector, then the expectation of the random vertex $v(U,P)$ is the Steiner point of P

$$s(P)=Ev(U,P) \tag{2}$$

from which it follows that

$$\langle s(P),x\rangle=E\langle v(U,P),x\rangle \qquad \text{for all } x\in\mathbf{E}^d. \tag{3}$$

We next define for a convex body K

$$\operatorname{sc}(K)=\bigcap_{\varepsilon>0}\overline{\{s(K')\,|\,\rho(K,K')<\varepsilon,\ K' \text{ finite dimensional convex body}\}}$$

(sc for "Steiner core") and note that standard methods immediately determine $\mathrm{sc}(K)$ to be a non-empty compact convex subset of K. Its behavior is in fact strongly dimension dependent and will suit our needs for the extension question.

THEOREM. *Let K be a convex body in $\mathbf{H}$. Then*

$$\mathrm{sc}(K) = \begin{cases} \{s(K)\} & \text{if } \mathbf{H} \text{ is finite dimensional,} \\ K & \text{if } \mathbf{H} \text{ is infinite dimensional.} \end{cases}$$

Note that the assertion for infinite dimensions is that any point of K is the limit of Steiner points of a convergent sequence of finite dimensional bodies and *a fortiori* continuous extension is impossible.

COROLLARY. *The Steiner point has no continuous extension to all convex bodies in infinite dimensional Hilbert spaces.*

PROOF OF THEOREM. The assertion in finite dimensions is merely the continuity of the Steiner point.

For infinite dimensions, in view of the comment preceding the theorem, it is enough to show that $h(u, \mathrm{sc}(K)) = h(u, K)$ for each $u \in \mathbf{H}$, $\|u\| = 1$. This is the case if, for each u and $\varepsilon > 0$, there is a sequence of finite dimensional bodies $\{K_N\}$, $\rho(K_N, K) < \varepsilon$, such that $\langle s(K_N), u\rangle \to h(u, K)$.

To that end, let $K \in \mathbf{K}$ and fix $u \in \mathbf{H}$, $\|u\| = 1$. For $\varepsilon > 0$, let $\{x_1, \dots, x_n\}$ be an $\varepsilon/2$-net for K with $\langle u, x_1\rangle = h(u, K)$. Locate N orthonormal vectors $e_1, \dots, e_N$ such that $\mathrm{span}\{e_1, \dots, e_N\} \perp \mathrm{span}\{u, x_1, \dots, x_n\}$. Set

$$K(n, N) = \mathrm{conv}\{x_1, \dots, x_n, y_1, \dots, y_{2^N}\}$$

where $\{y_j\}$ exhausts the possibilities for

$$x_1 + \frac{\varepsilon}{2\sqrt{N}} \sum_{i=1}^{N} \delta_i e_i, \qquad \delta_i = +1 \text{ or } -1.$$

Note that $K(n, N)$ is a convex polytope which satisfies

$$\rho(K, K(n, N)) \leq \rho(K, \mathrm{conv}\{x_1, \dots, x_n\}) + \rho(\mathrm{conv}\{x_1, \dots, x_n\}, K(n, N)) < \varepsilon$$

and

$$\|K(n, N)\| = \max\{\|x\| \mid x \in K(n, N)\} \leq \|K\| + \varepsilon.$$

As $N \to \infty$, it will be seen that $\langle s(K(n, N)), u\rangle \to h(u, K)$.

Let $U(n,N)$ be a random unit vector in $\text{span}\{u,x_1,\ldots,x_n,e_1,\ldots,e_N\}$ with decomposition $U(n,N)=U'(n,N)+U''(n,N)$, $U'(n,N)\in\text{span}\{u,x_1,\ldots,x_n\}$, $U''(n,N)\in\text{span}\{e_1,\ldots,e_N\}$. If $v(U(n,N),K(n,N))$, or more concisely $v(n,N)$, is a random vertex of $K(n,N)$, as discussed above, then

$$s(K(n,N)) = Ev(n,N)$$

and

$$\langle s(K(n,N)),u\rangle = E\langle v(n,N),u\rangle. \tag{4}$$

The sequence of random variables $\{\langle v(n,N),u\rangle\}_N$ is uniformly bounded in absolute value (by $\|K\|+\varepsilon$). Hence, a sufficient condition for the convergence of (4) to $h(u,K)$ is that the probability of the event

$$\langle v(n,N),u\rangle = h(u,K) = \langle u,x_1\rangle \tag{5}$$

tend to one. Note that (5) occurs if $v(n,N)\in\{y_1,\ldots,y_{2^N}\}$ and that this event is implied by

$$\max_j\langle y_j,U(n,N)\rangle > \max_i\langle x_i,U(n,N)\rangle. \tag{6}$$

Elementary estimates give

$$\max_i\langle x_i,U(n,N)\rangle \le \|U'(n,N)\|\cdot\max_i\|x_i\| \le \|U'(n,N)\|\cdot[\|K\|+\varepsilon]$$

and

$$\begin{aligned}\max_j\langle y_j,U(n,N)\rangle &= \frac{\varepsilon}{2\sqrt{N}}\sum_{k=1}^{N}|\langle e_k,U''(n,N)\rangle| + \langle x_1,U'(n,N)\rangle\\ &\ge \frac{\varepsilon}{2\sqrt{N}}\sum_{k=1}^{N}|\langle e_k,U''(n,N)\rangle| - \|U'(n,N)\|\cdot[\|K\|+\varepsilon]\end{aligned}$$

so that (6) and hence (5) occur if

$$\frac{\varepsilon}{2\sqrt{N}}\sum_{k=1}^{N}|\langle e_k,U''(n,N)\rangle| > 2\|U'(n,N)\|\cdot[\|K\|+\varepsilon]. \tag{7}$$

Appealing to [14], for instance, for the asymptotics of uniform spherical distributions provides that the right side of (7) tends to zero in probability and the left side tends to $\frac{1}{2}\varepsilon\cdot E|Z|$ for an $N(0,1)$ variable Z. Hence, (7) and (5) occur with probability tending to one as $N\to\infty$. This completes the proof.

4. Appendix

The exact form of the Lipschitz constant for the Steiner point does not seem to be readily accessible in the literature. We record it here for reference.

THEOREM. *In* $\mathbf{E}^d$,

$$\sup_{K \neq K'} \|s(K) - s(K')\| / \rho(K, K') = \frac{\Gamma\left(\frac{d}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2}\right)}.$$

PROOF. For an arbitrary unit vector e,

$$\langle e, s(K)\rangle = \left\langle e, d\int_{S^{d-1}} uh(u,K)\sigma(du)\right\rangle = d\int_{S^{d-1}} \langle e,u\rangle h(u,K)\sigma(du).$$

With the change of variables $x = \langle e, u\rangle$ and letting $\bar{h}(x,K)$ be the average of $h(u,K)$ over all u with $\langle e,u\rangle = x$, the expression becomes

$$\langle e, s(K)\rangle = d\int_{-1}^{+1} x\bar{h}(x,K)\frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d-1)/2)}(1-x^2)^{(d-3)/2}dx$$

$$= \frac{d\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d-1)/2)}\int_{-1}^{+1} x(1-x^2)^{(d-3)/2}\bar{h}(x,K)dx$$

and consequently

$$\langle e, s(K) - s(K')\rangle = \frac{d\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d-1)/2)}\int_{-1}^{+1} x(1-x^2)^{(d-3)/2}[\bar{h}(x,K) - \bar{h}(x,K')]dx.$$

Setting $e = [s(K) - s(K')]/\|s(K) - s(K')\|$, employing the bound

$$|\bar{h}(x,K) - \bar{h}(x,K')| \leq \rho(K,K'),$$

and performing an integration yield the asserted value. It can be approached arbitrarily closely by the following configuration. Fix $e \in S^{d-1}$ and R a large positive number. Let K be the convex hull of two hemispheres as given by

$$K = \mathrm{conv}\{R \cdot S^{d-1} \cap \{x \mid \langle x,e\rangle < 0\} \cup (R+1)\cdot S^{d-1} \cap \{x \mid \langle x,e\rangle \geq 0\}$$

and let $K' = -K$.

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